Maxwell-Dirac-Isomorphism, XII

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Maxwell-Dirac-isomorphism and spinor calculus.

In the last two papers [1] and [2] we have seen that covariant source-free electrodynamics

$$\left\{\begin{aligned} &\operatorname{rot}\boldsymbol{E}+\frac{\mu}{\varepsilon}\dot{\boldsymbol{H}}=0\\ &\operatorname{rot}\boldsymbol{H}-\frac{\varepsilon}{c}\dot{\boldsymbol{E}}=0\\ &\operatorname{div}\varepsilon\boldsymbol{E}=0\\ &\operatorname{div}\mu\boldsymbol{H}=0\end{aligned}\right\}_{\operatorname{div}\boldsymbol{E}=\operatorname{div}\boldsymbol{H}=0} \equiv \left\{\begin{aligned} &\operatorname{rot}\boldsymbol{E}+\frac{\mu}{c}\dot{\boldsymbol{H}}=0\\ &\operatorname{rot}\boldsymbol{H}-\frac{\varepsilon}{c}\dot{\boldsymbol{E}}=0\end{aligned}\right\}_{\substack{\operatorname{div}\boldsymbol{E}=\operatorname{div}\boldsymbol{H}=0\\ \boldsymbol{E}\perp\operatorname{grad}\varepsilon\\ \boldsymbol{H}\perp\operatorname{grad}\mu}}$$

and covariant wave mechanics

$$(\partial_\mu \gamma^\mu + q)\,\Psi = 0$$

posses the same amplitude equation

$$\underset{\text{(electrodynamics)}}{\left\{\left[\gamma\cdot\nabla+i\,\frac{\omega}{c}\begin{pmatrix}\varepsilon\,\mathbb{1} & \mathbb{0}\\ \mathbb{0} & \mu\,\mathbb{1}\end{pmatrix}\right]\Psi=0\right\}} \equiv \underset{\text{(wave mechanics)}}{\left\{\left[\gamma\cdot\nabla+i\,\frac{\omega}{c}\begin{pmatrix}\left(1-\dfrac{\Phi-m_0c^2}{\hbar\omega}\right)\mathbb{1} & \mathbb{0}\\ \mathbb{0} & \left(1-\dfrac{\Phi+m_0c^2}{\hbar\omega}\right)\mathbb{1}\end{pmatrix}\right]\Psi=0\right\}}. \tag{3}$$

The isomorphism (3), however, appears only if the harmonic time dependence

$$\Psi = \Psi\, e^{i\omega t} \tag{4}$$

can be assumed for the set of solutions. But the covariant time-dependent equations of the two theories differ in their differential structure (source-free electrodynamics is homogeneous, the Dirac-equation is inhomogeneous), and electrodynamics (1) can only be made equivalently inhomogeneous if the same time dependence (4) is assumed for both systems of equations.

The different structures of the time-dependent equations is the cause for the different behaviour under transformations of both theories, and the question arises of how the tensor theory (1) is related to the spinor theory (2). If one tries, in view of the isomorphism of the amplitudes, to bring into conformity the time-dependent equations too, one realizes: (2) can not be brought into accordance with (1) since the Dirac field is a spinor of odd rank [3]. On the other hand, (1) can easily be expressed by spinor equations, but they, of course, can not be identified with (2). Thus one obtains in a well-known manner the spinor correlated to the first Maxwell equation of (1) after a decomposition of the antisymmetric electromagnetic tensor h^{kl} into its self-dual part $h^{(\mathrm{d})\,kl}$ and its self-antidual part $h^{(\mathrm{a})\,kl}$ according to

$$h^{kl} = h^{(\mathrm{d})\,kl} + h^{(\mathrm{a})\,kl} \tag{5}$$

with

$$h^{(\mathrm{d})\,kl} = \tfrac{1}{2}\begin{pmatrix} 0 & H_3 - ic\varepsilon E^3 & -H_2 + ic\varepsilon E^2 & -c\varepsilon E^1 - iH_1\\ -H_3 + ic\varepsilon E^3 & 0 & H_1 - ic\varepsilon E^1 & -c\varepsilon E^2 - iH_2\\ H_2 - ic\varepsilon E^2 & -H_1 + ic\varepsilon E^1 & 0 & -c\varepsilon E^2 - iH_3\\ c\varepsilon E^1 + iH_1 & c\varepsilon E^2 + iH_2 & c\varepsilon E^3 + iH_3 & 0 \end{pmatrix} \tag{6}$$

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and

$$h^{(a)\,kl} = \frac{1}{2}\begin{pmatrix} 0 & H_3 + i c \varepsilon E^3 & -H_2 - i c \varepsilon E^2 & -c \varepsilon E^1 + i H_1 \\ -H_3 - i c \varepsilon E^3 & 0 & H_1 + i c \varepsilon E^1 & -c \varepsilon E^2 + i H_2 \\ H_2 + i c \varepsilon E^2 & -H_1 - i c \varepsilon E^1 & 0 & -c \varepsilon E^3 + i H_3 \\ c \varepsilon E^1 - i H_1 & c \varepsilon E^2 - i H_2 & c \varepsilon E^3 - i H_3 & 0 \end{pmatrix} \tag{7}$$

After introducing the symmetric spinors $h_{\dot{\varkappa}\dot{\lambda}}$ and $h^{\alpha\beta}$ through

$$\begin{aligned} h^{(d)\,kl} &= (\tfrac{1}{2})^2\, \sigma^{k\alpha\dot{\varkappa}}\, \sigma^{l\beta\dot{\lambda}}\, h_{\dot{\varkappa}\dot{\lambda}}\, \gamma_{\alpha\beta}, \\ h^{(a)\,kl} &= (\tfrac{1}{2})^2\, \sigma^{k}_{\alpha\dot{\varkappa}}\, \sigma^{l}_{\beta\dot{\lambda}}\, h^{\alpha\beta}\, \gamma^{\dot{\varkappa}\dot{\lambda}}, \end{aligned} \tag{8}$$

we get for the spinor components

$$\begin{aligned} h_{\dot{1}\dot{1}} &= -i\,(H_1 - i\,H_2) - c\,\varepsilon\,(E^1 - i\,E^2), \\ h_{\dot{1}\dot{2}} &= c\,\varepsilon\,E^3 + i\,H_3, \\ h_{\dot{2}\dot{2}} &= i\,(H_1 + i\,H_2) + c\,\varepsilon\,(E^1 - i\,E^2), \\ h^{11} &= -i\,(H_1 - i\,H_2) + c\,\varepsilon\,(E^1 - i\,E^2), \\ h^{12} &= -\,c\,\varepsilon\,E^3 + i\,H_3, \\ h^{22} &= -(H_1 + i\,H_2) - c\,\varepsilon\,(E^1 - i\,E^2). \end{aligned} \tag{9}$$

Here the expressions of [1] (17) appear again, they, however, do not make any easily recognizable sense, as could have been expected.

In conclusion it should therefore be said that the difference in the two time-dependent equations not only causes a different transformation behaviour of the two theories, but also that wave mechanics, as evident from [1] (17), possesses just one half of the manifold of solutions of electrodynamics. This lack in theory of wave mechanics has its origin in the largely arbitrary concept of the time-dependent Schrödinger equation, which just focusses on the elimination of the energy parameter. As we shall see later, this is the cause for the fact that the Dirac theory, among others, cannot bring about half-integer spin.

Since for concrete solutions the frequency conditions always follow from the amplitude equations, it is proved through the isomorphism (3) that electrodynamics possesses the hydrogen spectrum.

[1] H. Sallhofer, Z. Naturforsch. **41 a,** 468 (1986).
[2] H. Sallhofer, Z. Naturforsch. **41 a,** 1087 (1986).
[3] F. Cap, Fortschr. Physik **2,** 207 (1954).